A NOTE ON FIXED POINTS OF COMPLETELY POSITIVE MAPS

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July 2003

Given a Hilbert space $\mathcal{H}$ let $(x_t)_{t\in T}$ be a strongly (equivalently weakly) measurable field of operators in $\mathbb{B}(\mathcal{H})$ over a locally compact Hausdorff space T. Assume moreover that $\int_t x_t^* x_t \, d\mu(t) = \mathbf{1}$ and $\int_T x_t x_t^* \, d\mu(t) = e \le \mathbf{1}$ for some Borel measure μ on T, where the integrals are computed, say, by considering the corresponding sesquilinear forms, e.g. $(e\xi|\eta) = \int_T (x_t^*\xi | x_t^*\eta) \, d\mu(t)$ for ξ, η in $\mathcal{H}$.

This means in particular that the linear map

$$\Phi\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H}) \quad \text{given by} \quad \Phi(a) = \int_T x_t^* a x_t \, d\mu(t)$$

is a completely positive, unital map. Usually completely positive maps are presented by a sequence, i.e. by taking $T = \mathbb{N}$ and μ the counting measure, but there are cases where the continuous model may be preferred. Note that the condition $\int x_t x_t^* \, d\mu(t) \le \mathbf{1}$ puts a restraint on the type of completely positive maps to be described. It is most obviously satisfied if $x_t = x_t^*$ for all t in T.

If a is an element in $\mathbb{B}(\mathcal{H})$ commuting with the field $(x_t)_{t\in T}$, then evidently $\Phi(a) = a$. The surprising fact is that in many cases this is the only way fixed points can arise.

If M is a semi-finite von Neumann subalgebra of $\mathbb{B}(\mathcal{H})$ we say that an element a in M_+ is *finite*, if $\tau_i(a) < \infty$ for a separating family $\{\tau_i\}$ of normal, semi-finite traces on M.

Theorem. *With Φ and M as above, assume that $\Phi(M) \subset M$ and that a is a finite element in M_+ with $\Phi(a) \ge a$. Then $\Phi(a) = a$ and as a consequence $ax_t = x_t a$ for $\mu-$almost all t in T.*

Proof. Let τ be a normal, semi-finite trace on M with $\tau(a) < \infty$. Then

$$\begin{aligned}\tau(\Phi(a)) &= \int_T \tau(x_t^* a x_t) \, d\mu(t) = \int_T \tau(a^{1/2} x_t x_t^* a^{1/2}) \, d\mu(t) \\ &= \tau\left(a^{1/2}\left(\int_T x_t x_t^* \, d\mu(t)\right) a^{1/2}\right) = \tau(a^{1/2} e a^{1/2}) \le \tau(a) \, .\end{aligned} \tag{$*$}$$

2000 Mathematics Subject Classification Primary 46L10; Secondary 46L50, 46L60, 47C15.
Key words and phrases. Completely Positive Maps, Von Neumann Algebras, Fixed Points.

It follows that $\Phi(a)$ is a finite element of M_+. If now $\Phi(a) \geq a$ then $\Phi(a) - a$ is a finite element in M_+ and by $(*)$

$$0 \leq \tau(\Phi(a) - a) = \tau(\Phi(a)) - \tau(a) \leq 0.$$

Thus $\Phi(a) - a$ is annihilated by a separating family of semi-finite, normal traces on M, whence $\Phi(a) - a = 0$, as desired. This proves the first statement in the Theorem.

For each real ε define $f_\varepsilon(t) = t^2(1 - \varepsilon t)^{-1}$ on the open interval $]-|\varepsilon|^{-1}, |\varepsilon|^{-1}[$. It follows from the results of Bendat and Sherman, [**2**] or [**6**], that f_ε is an operator convex function. As shown in [**6**], see also [**7**], this implies that f_ε satisfies the *Jensen operator inequality*

$$f_\varepsilon\left(\sum_{k=1}^n x_k^* a x_k\right) \leq \sum_{k=1}^n x_k^* f_\varepsilon(a) x_k$$

for every finite set of operators (x_k) with $\sum x_k^* x_k \leq \mathbf{1}$. Since f_ε is operator continuous on any closed, bounded subinterval of $]-|\varepsilon|^{-1}, |\varepsilon|^{-1}[$ it follows by standard approximation arguments that we then also have

$$f_\varepsilon\left(\int_T x_t^* a x_t \, d\mu(t)\right) \leq \int_T x_t^* f_\varepsilon(a) x_t \, d\mu(t)$$

whenever $(x_t)_{t \in T}$ is a measurable operator field with $\int_T x_t^* x_t \, d\mu(t) \leq \mathbf{1}$.

In our case this means that

$$f_\varepsilon(a) = f_\varepsilon(\Phi(a)) \leq \Phi(f_\varepsilon(a)).$$

Since $f_\varepsilon(a)$ (for $|\varepsilon| < \|a\|^{-1}$) is a finite element in M_+ (dominated by λa, where $\lambda = \|a\|(1 - |\varepsilon|\|a\|)^{-1}$), it follows from the first part of the proof that actually $f_\varepsilon(a) = \Phi(f_\varepsilon(a))$. Expanding f_ε in a norm convergent series we therefore have

$$a^2 + \varepsilon a^3 + \varepsilon^2 a^4 + \ldots = \Phi(a^2) + \varepsilon\Phi(a^3) + \varepsilon^2\Phi(a^4) + \ldots$$

for every ε in a neighbourhood of zero, from which we conclude that $\Phi(a^n) = a^n$ for every n.

It follows that $\Phi(f(a)) = f(a)$ for every polynomial f, hence by continuity and approximation for every bounded measurable function f. In particular $\Phi(p) = p$ for every spectral projection p of a. Arguing as in [**4**, Lemma 3.3] this means that

$$0 = (\mathbf{1} - p)p(\mathbf{1} - p) = \int_T (\mathbf{1} - p)x_t^* p x_t(\mathbf{1} - p) \, d\mu(t),$$

whence $px_t(\mathbf{1} - p) = 0$ for almost all t in T. However, this computation is also valid for the spectral projection $\mathbf{1} - p$, so $(\mathbf{1} - p)x_t p = 0$ as well. We conclude that $x_t p = p x_t$ almost everywhere, and since this holds for every spectral projection, also $x_t a = a x_t$ almost everywhere, as desired. □

Corollary. *If $\Phi(a) = a$ and a^2 is finite in M, then $x_t a = a x_t$ for $\mu-$almost all t in T.*

Proof. Since $\Phi(a)^2 \leq \Phi(a^2)$ for every completely positive, unital map, the condition $\Phi(a) = a$ implies that $a^2 \leq \Phi(a^2)$. If now a^2 is finite we must have $\Phi(a^2) = a^2$ by the Theorem. Consequently $x_t p = p x_t$ for $\mu-$almost all t in T and every spectral projection p for a^2. Since a and a^2 have the same spectral family we conclude that $x_t a = a x_t$ almost everywhere. □

Remarks. With $T = \mathbb{N}$, $x_n = x_n^*$ and $M = \mathbb{B}(\mathcal{H})$ (so that a is a trace class operator) the result in the Theorem is known to phycisists as the Lüders Theorem, cf. [**10**] and [**3**]. It is shown in [**1**, Theorem 4.2] that the result may fail in general, i.e. without extra conditions on either a or (x_n). Indeed, if the von Neumann subalgebra A of $\mathbb{B}(\mathcal{H})$ generated by the family (x_n) is not injective then there is an operator a such that $\Phi(a) = a$, but $a \notin A'$, cf. [**1**, Theorem 3.6].

That the condition af finiteness on a, although sufficient, is not necessary is shown in [**3**, Proposition 1] and also in [**1**, Theorem 3.2]. Indeed, if a is an element in $\mathbb{B}(\mathcal{H})_+$ with a pure point spectrum that can be totally ordered in decreasing order, and if furthermore $x_t = x_t^*$ for all t in T, then $\Phi(a) = a$ (or just $\Phi(a) \geq a$) implies that $a x_t = x_t a$ for $\mu-$almost all t in T. The simple argument is reproduced below.

By assumption there is an orthogonal family (p_n) of projections with sum $\mathbf{1}$ such that $a = \sum \lambda_n p_n$ (strongly convergent sum), where $\lambda_1 > \lambda_2 > \ldots \geq 0$. If $\Phi(a) \geq a$ then for each unit vector ξ in the eigenspace $p_1(\mathcal{H})$ we have

$$\lambda_1 = (a\xi|\xi) \leq (\Phi(a)\xi|\xi) = \int_T (a x_t \xi | x_t \xi)\, d\mu(t) \leq \lambda_1 \int_T (x_t \xi | x_t \xi)\, d\mu(t) = \lambda_1,$$

since $\|a\| = \lambda_1$. Consequently $\lambda_1 \mathbf{1} - a \geq 0$ and

$$\int_T ((\lambda_1 \mathbf{1} - a) x_t \xi | x_t \xi)\, d\mu(t) = \lambda_1 - \lambda_1 = 0,$$

whence $a x_t \xi = \lambda_1 x_t \xi$ for $\mu-$almost all t in T. Thus $x_t p_1(\mathcal{H}) \subset p_1(\mathcal{H})$, which implies that $p_1 x_t = x_t p_1$ since $x_t = x_t^*$. Using that $\Phi(p_1) = p_1$ we pass to the operator $a_1 = a - \lambda_1 p_1$ in $\mathbb{B}(\mathcal{H})_+$, which again satisfies the condition $\Phi(a_1) \geq a_1$, and proceed by induction to show that $p_n x_t = x_t p_n$ for $\mu-$almost all t and every n, whence $a x_t = x_t a$, as desired.

This result, which contains the original Lüders Theorem, shows that $\Phi(a) = a$ implies that $a \in \{x_n\}'$ for every positive, compact operator. Making the generalizing from $\mathbb{B}(\mathcal{H})$ to a semi-finite factor M of type II with faithful, semi-finite trace τ, one could therefore hope that the Theorem would be valid (when $x_t = x_t^*$) not only for elements a in M_+ with $\tau(a) < \infty$ or $\tau(a^2) < \infty$, but for all positive elements in the norm closed ideal M_τ generated by the finite elements (or just the finite projections) in M.

References

[1] Alvaro Arias, Aurelian Gheondea & Stanley Gudder, *Fixed points of quantum operations*, Journal of Mathematical Physics **43** (2002), 5872–5881.

[2] Julius Bendat & Seymour Sherman, *Monotone and convex operator functions*, Transactions of the American Mathematical Society **79** (1955), 58–71.

[3] Paul Busch & Javed Singh, *Lüders theorem for unsharp quantum measurements*, Physical Letters A **249** (1998), 10–12.

[4] Ola Bratteli, Palle E.T. Jorgensen, Akitaka Kishimoto & Reinhard F. Werner, *Pure states on $\mathcal{O}_d$*, Journal of Operator Theory **43** (2000), 97–143.

[5] Frank Hansen, *Operator inequalities associated with Jensen's inequality*, "Survey on Classical Inequalities", editor T.M. Rassias, Kluwer Academic Publishers, 2000, pp. 67–98.

[6] Frank Hansen & Gert K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Mathematische Annalen **258** (1982), 229–241.

[7] Frank Hansen & Gert K. Pedersen, *Jensen's operator inequality*, Bulletin of the London Mathematical Society **35** (2003), 553–564.

[8] Richard V. Kadison & John R. Ringrose, *"Fundamentals of the Theory of Operator Algebras", vol I-II*, Academic Press, San Diego, 1986 (Reprinted by AMS in 1997).

[9] Fritz Kraus, *Über konvexe Matrixfunktionen*, Mathematische Zeitschrift **41** (1936), 18-42.

[10] G. Lüders, *Über die Zustandsänderung durch den Messprozess*, Annals of Physics **8** (1951), 322.

[11] Gert K. Pedersen, *"C^*-Algebras and their Automorphism Groups"*, LMS Monographs **14**, Academic Press, San Diego, 1979.

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